Principle of Duality

Any theorem of Boolean Algebra remains valid if '+' is interchanged with '*' and '0' is interchanged with '1' throughout the theorem.

Th → The following holds in a Boolean Algebra B,

1. Idempotent laws.

$$a+a=a \quad \& \quad a*a=a \quad \forall a \in B$$

Pf → First we shall prove that

$$a+a=a \quad \forall a \in B$$

We have,

$$a = a+0 = a+(a*a') \quad , \because 0 = a*a'$$

$$= (a+a)*(a+a') \quad , \text{dist}^{r}$$

$$= (a+a)*1$$

$$= a+a$$

i.e. $a = a+a$

Now to show →

$$a*a=a$$

We have,

$$a = a*1 = a*(a+a')$$

$$= (a*a)+(a*a')$$

$$= (a*a)+0$$

$$= a*a$$

$\therefore$ $a = a*a$

(Proved)

2. Boundedness Law:

$a+1=1$ & $a*0=0$ $\forall a \in B$

Pf→ First we shall show

$a+1=1$

we have

$$1 = a+a' = a+(a'*1), \quad \because a' = a'*1$$
$$= (a+a')*(a+1)$$
$$= 1*(a+1)$$
$$= a+1$$

$\therefore$ $\boxed{a+1=1}$

Since principle of duality holds in a Boolean Alg.,

$\therefore$ replace '+' by '*' & 1 by '0', we get—

$\boxed{a*0=0}$

(3) Absorption Laws:

$a+(a*b)=a$; $a*(a+b)=a$ $\forall a,b \in B$

(4) Associative Laws:

$(a+b)+c = (a+b)+c$; $(a*b)*c = a*(b*c)$ $\forall a,b,c \in B$

Thm Complement of an element 'a' in Boolean algebra B is unique. 5

Pf → Let 'a' be any element in a Boolean Algebra B. If possible, let x & y be two complements of a in B. Then-

$$a+x=1 \;;\; a*x=0 \quad \rightarrow (1)$$

&

$$a+y=1 \;;\; a*y=0 \quad \rightarrow (2)$$

To show $x=y$.

we have-

$$\begin{aligned}
x &= x*1 \\
&= x*(a+y) \\
&= (x*a)+(x*y) \\
&= 0+(x*y), && \text{from (1)} \\
&= x*y \\
&= (x*y)+0 \\
&= (x*y)+(a*y) && \text{, from (2)} \\
&= (x+a)*y, && \text{distr} \\
&= 1*y && \text{, from (1)} \\
&= y
\end{aligned}$$

∴ Complement of 'a' is unique.

(Proved)